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On the description of the phase transition in the Husimi–Temperley model

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Abstract. It is shown that the investigation of the magnetisation and the susceptibility of the finite-volume ferromagnetic Husimi–Temperley model reduces to the study of the nonlinear Burgers equation. The thermodynamic limit corresponds to vanishing diffusion coefficient and the resulting shock wave to the jump in the spontaneous magnetisation. The upper bound on the susceptibility of the finite system derived by the approximating Hamiltonian method is compared with the asymptotic form (as $N \to \infty$) of the susceptibility obtained from the solution of the Burgers equation.

1. Introduction

It is well known that phase transitions and spontaneous symmetry breaking may occur in the thermodynamic limit only. In the case of finite systems the macroscopic observables are smooth (moreover, real analytic) functions of the thermodynamic parameters: temperature, external fields, etc. The study of the mechanism of appearance of thermodynamic singularities is necessary not only for deeper understanding of phase transitions, but it can also provide some useful estimates of the diverging quantities in the thermodynamic limit as well. Such estimates turn out to be necessary, for example, in the proof of certain assertions about model systems (see Volovich *et al* 1973, Brankov *et al* 1977, 1979).

The purpose of this paper is to give the detailed description of the phase transition in the Ising model with infinitely long range, infinitesimally small interaction, the so called Husimi–Temperley model (see Husimi 1953, Temperley 1954). It will be shown that the occurrence of a phase transition in the thermodynamic limit is mathematically completely analogous to the formation of shock waves in nonlinear dispersive media, that is, to a problem extensively studied in nonlinear hydrodynamics (see Whitham 1974), nonlinear theory of condensed matter (e.g. Dynin 1979), chromatography (Yeroshenkova *et al* 1980), etc. By making use of this analogy we will obtain the asymptotic growth of the susceptibility of a finite system at the critical point with the increase of the size of the system and we will compare it with the upper bound found by Zagrebnov and Brankov (1973) with the aid of the approximating Hamiltonian method.

2. Susceptibility of a finite system: an upper bound

The Hamiltonian of the Husimi–Temperley model for a system of N spins ($s=\frac{1}{2}$) is

$$H_N = -\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i \qquad \{\sigma_i=\pm 1\} \tag{2.1}$$

where the ferromagnetic coupling constant $J>0$ and h is an external magnetic field. The corresponding approximating Hamiltonian (Brankov 1973) is

$$H_N^0(c) = -\sum_{i=1}^{N}(cJ+h)\sigma_i + \tfrac{1}{2}NJc^2. \tag{2.2}$$

Here $c\in\mathbb{R}$ is a trial c-number parameter. From the main theorem of the approximating Hamiltonian method (Bogolubov 1972, Bogolubov *et al* 1981) it follows that

$$0 \leqslant \inf_{c\in\mathbb{R}^1} f_N[H_N^0(c)] - f_N[H_N] \equiv \Delta_N(\beta,h) \leqslant \varepsilon_N(\beta,h). \tag{2.3}$$

Here $f_N[\ ]$ is the free energy per spin for the corresponding Hamiltonian:

$$f_N[\ ] = -(\beta N)^{-1}\ln \mathrm{Tr}\exp[-\beta(\ )], \tag{2.4}$$

β is the inverse temperature and $\varepsilon_N(\beta,h)\to 0$ as $N\to\infty$.

It is convenient to introduce the operator

$$s_N = N^{-1}\sum_{i=1}^{N}\sigma_i$$

with the aid of which we can write

$$\chi_N(\beta,h) = -\partial_h^2 f_N[H_N] = \beta N\langle(s_N-\langle s_N\rangle)^2\rangle \tag{2.5}$$

where $\langle\ \rangle$ denotes the thermodynamic average

$$\langle\ \rangle = \mathrm{Tr}\{e^{-\beta H_N}(\ )\}/\mathrm{Tr}\{e^{-\beta H_N}\}. \tag{2.6}$$

Hence for the susceptibility of the finite system we have the upper bound:

$$\chi_N(\beta,h) \leqslant \beta N\langle(s_N-\bar{c})^2\rangle = 2\beta N(\partial_J\Delta_N - \bar{c}\,\partial_h\Delta_N). \tag{2.7}$$

Here $\bar{c}=\bar{c}(\beta,h)$ corresponds to the point of absolute minimum of $f_N[H_N^0(c)]$ with respect to $c\in\mathbb{R}^1$ (see (2.3)), which is actually independent of N. Since $\partial_J^2 f_N[H_N]\leqslant 0$ we obtain

$$\partial_J^2\Delta_N \geqslant -\bar{c}^2\chi^0(\beta,h) \tag{2.8}$$

where $\chi^0(\beta,h)=\partial_h\bar{c}$ is the susceptibility of the approximating system. By making use of the equation for the order parameter $\bar{c}$ we get

$$0\leqslant \chi^0(\beta,h) = (1-\bar{c}^2)/[\beta^{-1}-J(1-\bar{c}^2)] \leqslant 3\beta^{-2}/2J(J\bar{c}+h)^2. \tag{2.9}$$

Hence

$$\bar{c}^2\chi^0(\beta,h)\leqslant 3/2\beta^2J^3; \tag{2.10}$$

therefore, the right-hand side of (2.8) is bounded uniformly in N from below.

In order to make use of this result for an upper bound on the susceptibility we note first that according to the Griffiths–Hurst–Sherman inequalities (Griffiths *et al* 1970)

$$\chi_N(\beta, h) \leq \chi_N(\beta, h=0). \tag{2.11}$$

By virtue of the thermodynamic equivalence of the Hamiltonians H_N and $H_N^0(c)$ the susceptibility $\chi_N(\beta, h=0)$ diverges as $N \to \infty$ only at the critical point $\beta_c = J^{-1}$ of the approximating system. Since $\bar{c}(\beta_c, 0) = 0$ and $|\partial_h \Delta_N| = |\bar{c} - \langle s_N \rangle| \leq 2$, from (2.7) and (2.11) we obtain

$$0 \leq \chi_N(\beta_c, h) \leq \chi_N(\beta_c, 0) \leq 2\beta_c N \partial_J \Delta_N(\beta_c, 0). \tag{2.12}$$

Now we recall the following lemma.

Lemma (Kolmogorov 1939, see also Bogolubov *et al* 1981). Let $\Delta_N(J)$ be a sequence of continuously differentiable functions, on the interval $\mathbb{I} = (0, \bar{J})$, having at any point $J \in [0, \bar{J})$ second-order derivatives from the right $\partial_J^2 \Delta_N(J+0)$ and let

(i) $|\Delta_N(J)| \leq \varepsilon_N(\mathbb{I}) \to 0 \qquad N \to \infty \qquad \forall J \in \mathbb{I}$

(ii) $\partial_J^2 \Delta_N(J+0) \geq -D(\mathbb{I}) > -\infty \qquad \forall J \in [0, \bar{J})$.

Then for $J \in [l_N, \bar{J} - l_N]$, $l_N = 2[\varepsilon_N(\mathbb{I})/D(\mathbb{I})]^{1/2}$ we have

$$|\partial_J \Delta_N(J)| \leq 2[\varepsilon_N(\mathbb{I}) D(\mathbb{I})]^{1/2}. \tag{2.13}$$

In the case under consideration all the conditions of the lemma are satisfied for the function $\Delta_N(\beta_c = J^{-1}, h=0) \equiv \Delta_N(J)$ with $\varepsilon_N(\mathbb{I}) = \varepsilon_N(J^{-1}, 0)$ and $D(\mathbb{I}) = \frac{3}{2}J$ (see (2.3), (2.8) and (2.11)). Therefore, the bound on the susceptibility takes the form:

$$\chi_N(\beta_c, h) \leq 4J^{-1}N[3\varepsilon_N(J^{-1}, 0)/2J]^{1/2}. \tag{2.14}$$

The approximating Hamiltonian method (Brankov 1973, Bogolubov *et al* 1981) gives an upper bound on $\varepsilon_N(\beta, h)$:

$$\varepsilon_N(\beta, h) \leq 2(\beta^{-1}J)^{1/2} N^{-1/2} \tag{2.15}$$

which implies the following bound on the susceptibility at the critical point:

$$\chi_N(\beta_c, h=0) \leq 4\sqrt{3}\beta_c N^{3/4}. \tag{2.16}$$

3. The magnetisation of a finite system and the Burgers equation

The asymptotic behaviour of $\chi_N(\beta, h)$ as $N \to \infty$ can be obtained by drawing an interesting analogy between the phase transition in model (2.1) and the appearance of shock waves in some dispersive nonlinear media. Indeed, in the case of the Husimi–Temperley model described by Hamiltonian (2.1) the average magnetisation per spin

$$\left\langle \frac{1}{N} \sum_{i=1}^{N} \sigma_i \right\rangle = m_N(x, t) \tag{3.1}$$

is a function of two variables: $t = \beta J$ and $x = -\beta h$. By differentiation of the explicit form of definition (3.1) (see (2.1) and (2.6)) with respect to t and x one obtains:

$$\partial_t m_N = \tfrac{1}{2}N(\langle s_N^3\rangle - \langle s_N^2\rangle\langle s_N\rangle)$$
$$\partial_x m_N = -N(\langle s_N^2\rangle - \langle s_N\rangle^2)$$
$$\partial_x^2 m_N = N^2(\langle s_N^3\rangle - 3\langle s_N^2\rangle\langle s_N\rangle + 2\langle s_N\rangle^3).$$

Therefore the function $m_N(t, x) \equiv \langle s_N\rangle$ obeys the Burgers equation (see e.g. Whitham 1974, ch 4):

$$\partial_t m_N + m_N \partial_x m_N = \tfrac{1}{2}N^{-1}\partial_x^2 m_N \tag{3.2}$$

with the initial condition

$$m_N(x, t=0) = -\tanh x. \tag{3.2a}$$

Equation (3.2) describes the evolution of perturbations in a dispersive medium in which the propagation velocity depends linearly on the amplitude and the perturbations disperse according to the diffusion law. In our case, the diffusion coefficient is inversely proportional to the number of spins and vanishes in the thermodynamic limit. This implies that the properties of the solutions of equation (3.2), i.e. the properties of the magnetisation, are qualitatively different for finite and infinite systems. The change in these properties can be traced explicitly.

By making use of the Cole–Hopf substitution, equation (3.2) can be solved explicitly:

$$m_N(x, t) = \int_{-\infty}^{\infty} d\eta \frac{x-\eta}{t} \exp[-NG(\eta; x, t)]\left(\int_{-\infty}^{\infty} d\eta \exp[-NG(\eta; x, t)]\right)^{-1}$$
$$G(\eta; x, t) = \int_0^{\eta} d\eta' \, m_N(\eta', t=0) + \frac{(x-\eta)^2}{2t} = -\ln\cosh\eta + \frac{(x-\eta)^2}{2t}. \tag{3.3}$$

Changing from the variable η to $c = (x-\eta)/t$ we obtain

$$\tilde{G}(c; x, t) = -\ln\cosh(ct - x) + \tfrac{1}{2}tc^2 = \beta f_\infty(c; h, \beta) + \ln 2$$

where

$$f_\infty(c; h, \beta) = f_N[H_N^0(c)] = -\beta^{-1}\ln 2\cosh(\beta Jc + \beta h) + \tfrac{1}{2}Jc^2 \tag{3.4}$$

is the free energy density for the approximating Hamiltonian (2.2).

From (3.2) and (3.3) it follows that with the increase of 'time' t the steepness of the solution $m_N(x, t)$ at the point $x = 0$ increases. For finite t, however, it always remains finite due to the presence of diffusion which smooths down the front (see e.g. Whitham 1974, ch 4). Since the diffusion coefficient is proportional to N^{-1}, the situation changes sharply in the thermodynamic limit. The formal passage to the limit $N \to \infty$ leads to the Burgers equation without a diffusion term. Now, due to the nonlinearity, with the increase of 'time' t the steepness of the solution at $x = 0$, which is proportional to the initial susceptibility, can become infinite at finite $t = t_c = \beta_c J$ with the subsequent formation of a shock wave, that is the function $m_\infty(x, t)$ becomes discontinuous with respect to x at $x = 0$ for $t > t_c$. This situation corresponds to the appearance of spontaneous magnetisation.

The above qualitative picture of the occurrence of a phase transition in the thermodynamic limit is confirmed in the next section by direct analysis of equation (3.3) and the derivative $\partial_h m_N = \chi_N$ which equals the susceptibility of the model.

4. Susceptibility of the Husimi–Temperley model: asymptotic form as $N \to \infty$

From (3.3) we see that the steepness of the front of the solution to the Burgers equation (3.2) coincides, up to a numeric factor, with the magnetic susceptibility of the Husimi–Temperley model

$$\chi_N(\beta, 0) = \beta N I_N^{(1)}(\beta)/I_N^{(2)}(\beta) \tag{4.1}$$

where

$$\begin{aligned} I_N^{(i)}(\beta) &= \int_{-\infty}^{\infty} \mathrm{d}c\, \varphi_i(c;\beta) \exp[-Nf_\infty(c; h=0, \beta)] \\ \varphi_1(c;\beta) &= c \tanh \beta J c \qquad \varphi_2(c;\beta) = 1. \end{aligned} \tag{4.2}$$

The large-N behaviour of the integrals $I_N^{(i)}(\beta)$ can be obtained with the aid of the general theory of Laplace integrals (see e.g. Dingle 1973, Fedoriuk 1977). From the relevant theorems it follows that

$$I_N^{(i)}(\beta) \sim N^{-1/2m} \exp[-Nf_\infty(\bar{c}; 0, \beta)] \sum_{k=0}^{\infty} N^{-k/m} a_k^{(i)}(\beta) \tag{4.3}$$

$$a_k^{(i)}(\beta) = -2\frac{(2m)^{2k}}{(2k)!}\Gamma\left(\frac{2k+1}{2m}\right)\{[g(c,\bar{c};\beta)\partial_c]^{2k}\varphi_i(c;\beta)g(c,\bar{c};\beta)\}|_{c=\bar{c}} \tag{4.4}$$

$$g(c,\bar{c};\beta) = [\beta f_\infty(c;0,\beta) - \beta f_\infty(\bar{c};0,\beta)]^{(2m-1)/2m}/[-\beta\partial_c f_\infty(c;0,\beta)]. \tag{4.5}$$

The point $c = \bar{c}$ is determined by the condition for the absolute minimum of function (3.4)

$$f_\infty(\bar{c};0,\beta) = \min_c f_\infty(c;0,\beta). \tag{4.6}$$

In the case when $f_\infty(c;0,\beta)$ reaches absolute minimum at more than one point (this happens when $\beta > \beta_c = J^{-1}$) the right-hand side of (4.4) has to be summed over the contributions from each such point. From (4.5) we obtain for $c = \bar{c}$:

$$g(c,\bar{c};\beta) = -\frac{1}{2m}\left(\frac{\beta}{(2m)!}f_\infty^{(2m)}(\bar{c};0,\beta)\right)^{-1/2m}. \tag{4.7}$$

Here the Taylor series expansions of $f_\infty(c;0,\beta)$ and $f'_\infty(c;0,\beta)$ about $c = \bar{c}$ have been used under the conditions

$$f_\infty^{(j)}(\bar{c};0,\beta) = 0 \qquad 1 \leqslant j \leqslant 2m-1 \qquad f_\infty^{(2m)}(\bar{c};0,\beta) \neq 0. \tag{4.8}$$

It suffices to consider only the first two terms in the series (4.3), namely those with $k = 0$ and $k = 1$:

$$\begin{aligned} a_0^{(i)}(\beta) &= -2\Gamma(\tfrac{1}{2}m^{-1})\varphi_i(\bar{c};\beta)g(\bar{c},\bar{c};\beta) \\ a_1^{(i)}(\beta) &= -4m^2\Gamma(\tfrac{3}{2}m^{-1})\{g^3(\bar{c},\bar{c};\beta)\varphi_i''(\bar{c};\beta) + 3g^2(\bar{c},\bar{c};\beta)g'(\bar{c},\bar{c};\beta)\varphi_i'(\bar{c};\beta) \\ &\quad + g^2(\bar{c},\bar{c};\beta)g''(\bar{c},\bar{c};\beta)\varphi_i(\bar{c};\beta) + g(\bar{c},\bar{c};\beta)[g'(\bar{c},\bar{c};\beta)]^2\varphi_i(\bar{c};\beta)\}. \end{aligned} \tag{4.9}$$

There are three cases to be distinguished.

(a) *At the critical point.* $\beta=\beta_c=J^{-1}$, $h=0$. Then from (3.4) and (4.6) we obtain $\bar{c}(\beta_c, 0)=0$ and

$$f'_\infty(0;0,\beta_c)=f''_\infty(0;0,\beta_c)=f'''_\infty(0;0,\beta_c)=0$$
$$\beta_c f''''_\infty(0;0,\beta_c)=2. \tag{4.10}$$

Therefore, in this case $m=2$ (see (4.8)) and

$$g(0,0;\beta_c)=-\tfrac{1}{4}\{\tfrac{1}{24}\beta_c f''''_\infty(0;0,\beta_c)\}^{-1/4}=-\tfrac{1}{4}(12)^{1/4}. \tag{4.11}$$

From the explicit expression (3.4) we find for $h=0$, $\beta=\beta_c$:

$$\beta_c f_\infty(c;0,\beta_c)=-\ln\cosh c+\tfrac{1}{2}c^2=\tfrac{1}{12}c^4-\tfrac{1}{45}c^6+\ldots$$
$$\beta_c f'_\infty(c;0,\beta_c)=-\tanh c+c=\tfrac{1}{3}c^3-\tfrac{2}{15}c^5+\ldots.$$

Hence

$$g(c,0;\beta_c)=g(0,0;\beta_c)(1+\tfrac{1}{5}c^2+\ldots)$$
$$g'(0,0;\beta_c)=0 \qquad g''(0,0;\beta_c)=\tfrac{2}{5}g(0,0;\beta_c). \tag{4.12}$$

Further, from (4.2) we obtain

$$\varphi_1(0;\beta_c)=\varphi'_1(0;\beta_c)=0 \qquad \varphi''_1(0;\beta_c)=2$$
$$\varphi_2(0;\beta_c)=1 \qquad \varphi'_2(0;\beta_c)=\varphi''_2(0;\beta_c)=0. \tag{4.13}$$

By inserting (4.12) and (4.13) in (4.9) we find

$$a_0^{(1)}(\beta_c)=0 \qquad a_0^{(2)}(\beta_c)=\tfrac{1}{2}(12)^{1/4}\Gamma(\tfrac{1}{4})$$
$$a_1^{(1)}(\beta_c)=\tfrac{1}{2}(12)^{3/4}\Gamma(\tfrac{3}{4}) \qquad a_1^{(2)}(\beta_c)=\tfrac{1}{10}(12)^{3/4}\Gamma(\tfrac{3}{4}).$$

Therefore, the asymptotic expansion (4.3) takes the form

$$I_N^{(1)}(\beta_c)\sim N^{-1/4}[\tfrac{1}{2}(12)^{3/4}\Gamma(\tfrac{3}{4})N^{-1/2}+\ldots]$$
$$I_N^{(2)}(\beta_c)\sim N^{-1/4}[\tfrac{1}{2}(12)^{1/4}\Gamma(\tfrac{1}{4})+\tfrac{1}{10}(12)^{3/4}N^{-1/2}+\ldots]. \tag{4.14}$$

Finally, the asymptotic form of the magnetic susceptibility (4.1) at the critical point is (compare with (2.16))

$$\chi_N(\beta_c,0)\sim\sqrt{12}(\Gamma(\tfrac{3}{4})/\Gamma(\tfrac{1}{4}))\beta_c N^{1/2}=1.17083\ldots\beta_c N^{1/2}. \tag{4.15}$$

(b) *Below the critical point.* $\beta>\beta_c$, $h=0$. In this case, function (3.4) reaches a minimum at two points: $c=\pm|\bar{c}|$. Now

$$\beta f''_\infty(c;0,\beta)|_{c=\pm|\bar{c}|}=\beta J(1-\beta J/\cosh^2\beta J\bar{c})\neq 0$$

and $m=1$ in (4.3)–(4.8). Therefore

$$g(\bar{c},\bar{c};\beta)=g(-\bar{c},-\bar{c};\beta)=-[2\beta f''_\infty(\bar{c};0,\beta)]^{-1/2}$$
$$\varphi_1(\pm\bar{c};\beta)=\bar{c}^2. \tag{4.16}$$

Hence, we have

$$a_0^{(1)}(\beta)=-2\Gamma(\tfrac{1}{2})\bar{c}^2 g(\bar{c},\bar{c};\beta) \qquad a_0^{(2)}(\beta)=-2\Gamma(\tfrac{1}{2})g(\bar{c},\bar{c};\beta). \tag{4.17}$$

As the contributions in (4.4) of both points of minimum $c=\pm|\bar{c}|$ are equal, as follows from (4.16) and (4.17), from (4.1)–(4.5) we obtain for $N\to\infty$:

$$\chi_N(\beta,0)\sim\beta\bar{c}^2 N \qquad \beta>\beta_c. \tag{4.18}$$

(c) *Above the critical point.* $\beta < \beta_c$, $h = 0$. There is a unique point $\bar{c} = 0$ of the minimum of function (3.4) and

$$\beta f''_\infty(0; 0, \beta) = \beta J(1 - \beta J) > 0.$$

Therefore $m = 1$ in (4.3)–(4.8) and

$$g(0, 0; \beta) = -[2\beta J(1 - \beta J)]^{-1/2}.$$

Now from (4.4) we have

$$a_0^{(1)}(\beta) = 0 \qquad a_0^{(2)}(\beta) = -2\Gamma(\tfrac{1}{2}) g(0, 0; \beta).$$

Since $a_0^{(1)}(\beta) = 0$ it is necessary to find $a_1^{(1)}(\beta)$, but (see (4.2))

$$\varphi_1(0; \beta) = \varphi_1'(0; \beta) = 0 \qquad \varphi_1''(0; \beta) = 2\beta J.$$

Therefore

$$a_1^{(1)}(\beta) = -4\Gamma(\tfrac{3}{2}) \varphi_1''(0; \beta) g^3(0, 0; \beta)$$

and the asymptotic form of the susceptibility in this case is

$$\chi_N(\beta, 0) \sim \beta/(1 - \beta J) = \chi_\infty(\beta, 0) \qquad \beta < \beta_c. \tag{4.19}$$

5. Concluding remarks

First we note that the results of the preceding section agree with the predictions of § 3. Indeed, for small 'times' $t < t_c$ the evolution of the initial condition (3.2a) in the nonlinear medium described by equation (3.2) is such that the steepness of the front at $x = 0$ is finite, including the limiting case $N \to \infty$. At the moment $t = t_c$ the steepness at $x = 0$ increases to infinity with $N \to \infty$, i.e. with the decrease to zero of the diffusion coefficient. Further, in the limit of vanishing diffusion, the formation of a shock wave takes place for $t > t_c$, which corresponds to the jump in the spontaneous magnetisation with the change of the magnetic field ($x = -\beta h$) across the point $x = 0$. We should note that in this case the quantity (4.18) is not the initial magnetic susceptibility of the Husimi–Temperley system, since the latter is defined as

$$\chi_\infty(\beta, h = 0) = \lim_{h \to 0^+} \lim_{N \to \infty} \chi_N(\beta, h). \tag{5.1}$$

This quantity, in contrast to (4.18), is finite for $\beta > \beta_c$ and coincides with the susceptibility defined by the approximating Hamiltonian method (Bogolubov *et al* 1981). Definition (5.1), or the analogous one for $h \to 0^-$, physically reflects the fact that the limiting Gibbs distribution, which is concentrated at the points $\pm|\bar{c}(\beta, 0)|$, is non-ergodic and only one of its ergodic components should be selected.

In conclusion we mention that the comparison of the asymptotic form (4.15) and the upper bound (2.16) indicates that the latter is a large overestimation. Since (2.16) follows from (2.14) and (2.15), the upper bound on the susceptibility can be improved by improving the bound (2.3). For example, the Laplace method (see Moschchinsky and Fedyanin 1977) or the direct evaluation of the partition function with the use of Stirling's formula (see Kittel and Shore 1965, Scharf 1972) result in

$$\varepsilon_N(\beta, h) \sim \beta^{-1} J N^{-1} \ln(N/2 + 1). \tag{5.2}$$

From (5.2) and (2.14) an upper bound on the susceptibility follows which is much closer to the asymptotic form (4.15). Equations (3.3) appear, in particular, in the study of the model with the use of the well known integral representation for an exponent of a quadratic form (see e.g. Kac 1968).

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